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Complemented Subspaces and Interpolation Properties in Spaces of Polynomials

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Let X be a Banach space whose dual X^* has type $p \in (1, 2]$. If m is an integer greater than $p/(p-1)$ and (x_n) is a seminormalized sequence weakly convergent to zero, there is a subsequence (y_n) of (x_n) such that, for each element (a_n) of l_∞, there is an m-homogeneous continuous polynomial P on X with $P(y_n) = a_n$, $n = 1, 2, \ldots$. Some interpolation and complementation properties are also given in $\mathscr{P}(^m l_p)$, for $m < p$, as well as in other spaces of polynomials and multilinear functionals.

The linear spaces we shall use in this paper are all assumed to be defined over the field K of real or complex numbers. $\mathbb{N}$ will denote the set of positive integers. For a set B, $|B|$ denotes its cardinal number.

If X is a Banach space, then X^* and X^{**} represent its conjugate and second conjugate, respectively. We identify X with a subspace of X^{**} by means of the canonical embedding. Otherwise stated, $B(X)$ is the closed unit ball of X; $\|\cdot\|_X$ is the norm on X, when no confusion occurs $\|\cdot\|$ will be also used. For $x \in X$ and $u \in X^*$, $\langle x, u \rangle$ means $u(x)$. If (x_n) is a sequence in X, $[x_n]$ denotes the closed linear span of (x_n). We say that (x_n) is seminormalized if there are $0 < h < k < \infty$ such that $h \leq \|x_n\| \leq k$, $n = 1, 2, \ldots$. If the sequence (x_n) is basic, (x_n^*) will be its associated sequence of functionals in $[x_n]^*$. We say that X is an Asplund space whenever each separable closed subspace has separable conjugate, or, equivalently, X^* has the Radon–Nikodym Property.

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The l_p spaces we use here correspond to $1 \leq p < \infty$; if $p = \infty$, we put explicitly l_∞.

If $X_1, X_2, \ldots, X_m$ are Banach spaces, we take as closed unit ball of $X_1 \times X_2 \times \cdots \times X_m$ the product $B(X_1) \times B(X_2) \times \cdots \times B(X_m)$. $\mathscr{L}(X_1, X_2, \ldots, X_m)$ denotes the space of all m-linear continuous functionals on $X_1 \times X_2 \times \cdots \times X_m$, with the usual norm. If $x_j^* \in X_j^*$, $j = 1, 2, \ldots, m$,

$$x_1^* \otimes x_2^* \otimes \cdots \otimes x_m^* \tag{1}$$

is the element of $\mathscr{L}(X_1, X_2, \ldots, X_m)$ such that, if $x_j \in X_j$, $j = 1, 2, \ldots, m$,

$$x_1^* \otimes x_2^* \otimes \cdots \otimes x_m^*(x_1, x_2, \ldots, x_m) = x_1^*(x_1) x_2^*(x_2) \ldots x_m^*(x_m).$$

When $X = X_1 = X_2 = \cdots = X_m$, we write $\mathscr{L}(^m X)$ instead of $\mathscr{L}(X_1, X_2, \ldots, X_m)$. $\mathscr{L}_s(^m X)$ will be the subspace of $\mathscr{L}(^m X)$ formed by the symmetric functionals. By taking the restrictions to the diagonal of the elements of $\mathscr{L}(^m x)$ we obtain a linear space represented by $\mathscr{P}(^m X)$. If $P \in \mathscr{P}(^m X)$, we say that P is an m-homogeneous continuous polynomial on X; thus, there is an element f of $\mathscr{L}(^m X)$ such that

$$P(x) := f\left(x, x, \overset{(m)}{\ldots}, x\right)$$

We define

$$\|P\| := \sup\{|P(x)| : \|x\| \leq 1\}$$

and assume that $\mathscr{P}(^m X)$ is endowed with this norm. By means of a polarization formula we may associate in a unique way to each polynomial P a symmetric element g of $\mathscr{L}(^m X)$ such that

$$g\left(x, \overset{(m)}{\ldots}, x\right) = P(x), \qquad x \in X.$$

As for tensor products, we use the notation of [11], extend here for tensor products of more than two factors.

LEMMA 1. *Let $X_1, X_2, \ldots, X_n$ be linear spaces. Let f be an n-linear functional defined in $X_1 \times X_2 \times \cdots \times X_n$. Given the vectors x_{jr} in X_j, $j = 1, 2, \ldots, n$, $r = 1, 2, \ldots, m$, there are numbers $\gamma_{jr} \in K$, $|\gamma_{jr}| = 1$, $j = 1, 2, \ldots, n$, $r = 1, 2, \ldots, m$, such that*

$$\left| \sum_{r=1}^{m} f(x_{1r}, x_{2r}, \ldots, x_{nr}) \right| \leq \left| f\left(\sum_{r=1}^{m} \gamma_{1r} x_{1r}, \sum_{r=1}^{m} \gamma_{2r} x_{2r}, \ldots, \sum_{r=1}^{m} \gamma_{nr} x_{nr} \right) \right|.$$

Proof. For $n = 1$, we have an equality by taking $\gamma_{1r} = 1$, $r = 1, 2, \ldots, m$. Following an induction process, let us assume that the property holds for all p-linear functionals on $X_1 \times X_2 \times \cdots \times X_p$, for a positive integer $p < n$, and we show that it still holds for $p+1$. If $m = 1$, we have nothing to prove. Assume then that $m > 1$. We put $\eta_{p1} := 1$ and choose $\gamma_{(p+1)1}$ in K such that $|\gamma_{(p+1)1}| = 1$ and

$$f(x_{11}, x_{21}, \ldots, x_{(p-1)1}, \eta_{p1} x_{p1}, \gamma_{(p+1)1} x_{(p+1)1}) \geq 0.$$

Proceeding by recurrence, suppose that, for a positive integer $s < m$, we have chosen in K

$$n_{pr} \text{ and } \gamma_{(p+1)r}, \qquad \text{with } |\eta_{pr}| = |\gamma_{(p+1)r}| = 1, \qquad r = 1, 2, \ldots, s,$$

such that

$$f(x_{1r}, x_{2r}, \ldots, x_{(p-1)r}, \eta_{pr} x_{pr}, \gamma_{(p+1)r} x_{(p+1)r}) \geq 0, \qquad r = 1, 2, \ldots, s, \tag{2}$$

$$\sum_{1 \leq j \neq r \leq s} \mathscr{R}e\big(f(x_{1j}, x_{2j}, \ldots, x_{(p-1)j}, \eta_{pj} x_{pj}, \gamma_{(p+1)r} x_{(p+1)r})\big) \geq 0. \tag{3}$$

We find $\delta_{(p+1)(s+1)}$ in K, $|\delta_{(p+1)(s+1)}| = 1$, such that

$$f\big(x_{1(s+1)}, x_{2(s+1)}, \ldots, x_{p(s+1)}, \delta_{(p+1)(s+1)} x_{(p+1)(s+1)}\big) \geq 0.$$

If the real part of

$$\sum_{r=1}^{s} f\big(x_{1r}, x_{2r}, \ldots, x_{(p-1)r}, \eta_{pr} x_{pr}, \delta_{(p+1)(s+1)} x_{(p+1)(s+1)}\big) + \sum_{r=1}^{s} f(x_{1(s+1)}, x_{2(s+1)}, \ldots, x_{p(s+1)}, \gamma_{(p+1)r} x_{(p+1)r})$$

is greater than or equal to zero, we write

$$\eta_{p(s+1)} := 1, \qquad \gamma_{(p+1)(s+1)} := \delta_{(p+1)(s+1)},$$

and if it is less than zero, we put

$$\eta_{p(s+1)} := -1, \qquad \gamma_{(p+1)(s+1)} := -\delta_{(p+1)(s+1)}.$$

Then we have that (2) and (3) are yet satisfied replacing s by $s+1$. Consequently, (2) and (3) hold for m. Hence,

$$
\begin{aligned}
&\left|\sum_{r=1}^{m} f(x_{1r}, x_{2r}, \ldots, x_{(p+1)r})\right| \\
&\qquad \leq \sum_{r=1}^{m} f(x_{1r}, x_{2r}, \ldots, x_{(p-1)r}, \eta_{pr} x_{pr}, \gamma_{(p+1)r} x_{(p+1)r}) \\
&\qquad \leq \left|\sum_{1\leq j, r\leq m} f(x_{1r}, x_{2r}, \ldots, x_{(p-1)r}, \eta_{pr} x_{pr}, \gamma_{(p+1)j} x_{(p+1)j})\right| \\
&\qquad = \left|\sum_{r=1}^{m} f\left(x_{1r}, x_{2r}, \ldots, x_{(p-1)r}, \eta_{pr} x_{pr}, \sum_{j=1}^{m} \gamma_{(p+1)j} x_{(p+1)j}\right)\right|.
\end{aligned}
$$

We now consider the p-linear functional on $X_1 \times X_2 \times \cdots \times X_p$ given by

$$
f\left(\cdot, \cdot, \ldots, \cdot, \sum_{j=1}^{m} \gamma_{(p+1)j} x_{(p+1)j}\right).
$$

Then, by our induction hypothesis, given the vectors

$$
x_{1r}, x_{2r}, \ldots, x_{(p-1)r}, \eta_{pr} x_{pr}, \qquad r = 1, 2, \ldots, m,
$$

there are $\gamma_{jp} \in K$, $|\gamma_{jp}| = 1$, $j = 1, 2, \ldots, p$, such that

$$
\begin{aligned}
&\left|\sum_{r=1}^{m} f\left(x_{1r}, x_{2r}, \ldots, x_{(p-1)r}, \eta_{pr} x_{pr}, \sum_{j=1}^{m} \gamma_{(p+1)j} x_{(p+1)j}\right)\right| \\
&\qquad \leq \left| f\left(\sum_{r=1}^{m} \gamma_{1r} x_{1r}, \sum_{r=1}^{m} \gamma_{2r} x_{2r}, \ldots, \sum_{r=1}^{m} \gamma_{(p-1)r} x_{(p-1)r},\right.\right. \\
&\qquad\qquad \left.\left. \sum_{r=1}^{m} \gamma_{pr} x_{pr}, \sum_{r=1}^{m} \gamma_{(p+1)r} x_{(p+1)r}\right)\right|.
\end{aligned}
$$

Thus,

$$
\begin{aligned}
&\left|\sum_{r=1}^{m} f(x_{1r}, x_{2r}, \ldots, x_{(p+1)r})\right| \\
&\qquad \leq \left| f\left(\sum_{r=1}^{m} \gamma_{1r} x_{1r}, \sum_{r=1}^{m} \gamma_{2r} x_{2r}, \ldots, \sum_{r=1}^{m} \gamma_{(p+1)r} x_{(p+1)r}\right)\right|,
\end{aligned}
$$

and the result follows. Q.E.D.

THEOREM 1. *Given $l_{p_1}, l_{p_2}, \ldots, l_{p_m}$ with*

$$\frac{1}{p_1} + \frac{1}{p_2} + \cdots + \frac{1}{p_m} = \frac{1}{r}, \qquad m > 1,\, r \geq 1,$$

let (e_{jn}) be a Schauder basis in l_{p_j} equivalent to the unit vector basis of this space, $j = 1, 2, \ldots, m$. If Y is the closed linear hull of

$$(e_{1n} \otimes e_{2n} \otimes \cdots \otimes e_{mn}) \tag{4}$$

in $X := l_{p_1} \tilde{\otimes}_\pi l_{p_2} \tilde{\otimes}_\pi \cdots \tilde{\otimes}_\pi l_{p_m}$, then Y is isomorphic to l_r, has a topological complement, and (4) is a Schauder basis for Y equivalent to the unit basis of l_r.

Proof. We do first the proof for $m = 2$. If $(a_n) \in l_r$, we consider the series

$$\sum_{n=1}^{\infty} a_n e_{1n} \otimes e_{2n} \tag{5}$$

in X. For a given $\epsilon > 0$, we find a positive integer n_0 such that, if $n_0 \leq q \leq s$,

$$\sum_{n=q}^{s} |a_n|^r < \epsilon^r.$$

We get an element x^* in X^*, $\|x^*\| = 1$, so that

$$\left\| \sum_{n=q}^{s} a_n e_{1n} \otimes e_{2n} \right\| = \left\langle \sum_{n=q}^{s} a_n e_{1n} \otimes e_{2n}, x^* \right\rangle.$$

Then, if B is the bilinear functional associated to x^* and α_n, $q \leq n \leq s$, are certain elements of K with $|\alpha_n| = 1$, we have that

$$\left\| \sum_{n=q}^{s} a_n e_{1n} \otimes e_{2n} \right\| = \sum_{n=q}^{s} a_n B(e_{1n}, e_{2n}) = \sum_{n=q}^{s} \alpha_n |a_n|^{r/p_1} |a_n|^{r/p_2} B(e_{1n}, e_{2n}).$$

We apply our former lemma to obtain $\beta_n,\ \gamma_n \in K,\ |\beta_n| = |\gamma_n| = 1,\ q \leq n \leq s$, such that

$$\begin{aligned}
\left\| \sum_{n=q}^{s} a_n e_{1n} \otimes e_{2n} \right\| &\leq \left| B\left(\sum_{n=q}^{s} \beta_n |a_n|^{r/p_1} e_{1n},\ \sum_{n=q}^{s} \gamma_n |a_n|^{r/p_2} e_{2n} \right) \right| \\
&\leq \left\| \sum_{n=q}^{s} \beta_n |a_n|^{r/p_1} e_{1n} \right\|_{l_{p_1}} \cdot \left\| \sum_{n=q}^{s} \gamma_n |a_n|^{r/p_2} e_{2n} \right\|_{l_{p_2}} \\
&= \left(\sum_{n=q}^{2} |a_n|^r \right)^{1/p_1} \cdot \left(\sum_{n=q}^{s} |a_n|^r \right)^{1/p_2} \\
&= \left(\sum_{n=q}^{s} |a_n|^r \right)^{1/r} < \epsilon,
\end{aligned}$$

hence it follows that the series (5) converges in X. Now, let

$$Z := \left\{ \sum_{n=1}^{\infty} a_n e_{1n} \otimes e_{2n} \colon (a_n) \in l_r \right\}.$$

It follows that Z is a linear subspace of X, which we assume endowed with the norm induced by that of X. We write

$$A := \{ x \otimes y \colon x \in l_{p_1},\ y \in l_{p_2} \}.$$

We have that $l_{p_1} \otimes l_{p_2}$ is the linear hull of A. If u belongs to $l_{p_1} \otimes l_{p_2}$, we define Tu as the sequence in K, $(\langle u, e_{1n}^* \otimes e_{2n}^* \rangle)$. If we fix x in l_{p_1} and y in l_{p_2}, we then have

$$x = \sum_{n=1}^{\infty} b_n e_{1n}, \qquad y = \sum_{n=1}^{\infty} c_n e_{2n},$$

hence

$$\langle x \otimes y, e_{1n}^* \otimes e_{2n}^* \rangle = b_n c_n.$$

Now, since p_1/r and p_2/r are conjugate numbers, Hölder's inequality yields

$$\sum_{n=1}^{\infty} |b_n c_n|^r \leq \left(\sum_{n=1}^{\infty} \left(|b_n|^r \right)^{p_1/r} \right)^{r/p_1} \left(\sum_{n=1}^{\infty} \left(|c_n|^r \right)^{p_2/r} \right)^{r/p_2},$$

and therefore

$$\left(\sum_{n=1}^{\infty}|b_n c_n|^r\right)^{1/r} \le \left(\sum_{n=1}^{\infty}\left(|b_n|^{p_1}\right)\right)^{1/p_1}\left(\sum_{n=1}^{\infty}\left(|c_n|^{p_2}\right)\right)^{1/p_2}, \tag{6}$$

thus T is a linear operator from $l_{p_1} \otimes l_{p_2}$ into l_r. Besides, if (a_n) is a given element of l_r, we find $\alpha_n \in K$, $|\alpha_n| = 1$, such that $\alpha_n|a_n| = a_n$, $n = 1, 2, \ldots$. Then

$$x_1 := \sum_{n=1}^{\infty} \alpha_n|a_n|^{r/p_1} e_{1n} \in l_{p_1}, \qquad y_1 := \sum_{n=1}^{\infty} |a_n|^{r/p_2} e_{2n} \in l_{p_2}$$

and $T(x_1 \otimes y_1) = (a_n)$, from where we deduce that

$$T\colon l_{p_1} \otimes_\pi l_{p_2} \to l_r \tag{7}$$

is an onto linear map. We see that T is continuous. Let B_r denote the closed unit ball of l_r. It follows then that $T^{-1}(B_r)$ is an absolutely convex subset of $l_{p_1} \otimes_\pi l_{p_2}$. If we assume that A is provided with the topology induced by that of $l_{p_1} \otimes_\pi l_{p_2}$, we have by (6) that $T_{|A}\colon A \to l_r$ is continuous at the origin, consequently $T^{-1}(B_r) \cap A$ is a zero-neighborhood in A and hence there is $h > 0$ such hat

$$M := \{x \otimes y\colon \|x\| \le h, \|y\| \le h\}$$

is contained in $T^{-1}(B_r) \cap A$. The absolutely convex hull of M in $l_{p_1} \otimes_\pi l_{p_2}$ is a zero-neighborhood in this space and it is contained in $T^{-1}(B_r)$, which gives continuity for (7). We extend T continuously to obtain a continuous linear operator

$$S\colon X \to l_r.$$

For a given w in X, we find a sequence (w_n) in $l_{p_1} \otimes_\pi l_{p_2}$ which converges to w in X. Then (Sw_n) converges to an element of l_r which is clearly $(\langle w, e_{1n}^* \otimes e_{2n}^* \rangle)$. Thus, for every element w of X, $(\langle w, e_{1n}^* \otimes e_{2n}^* \rangle)$ belongs to l_r. In $\mathbb{N} \times \mathbb{N}$ one can define an order relation $\le$ so that

$$\{e_{1n} \otimes e_{2m}\colon (n, m) \in \mathbb{N} \times \mathbb{N}, \le\} \tag{8}$$

is a Schauder basis of X with $(e_{1n} \otimes e_{2n})$ as a subsequence of this basis [9]. Hence, the elements of Z may be regarded as those vectors in X whose series expansion with respect to the basis (8) has zero entries in the coefficients corresponding to the terms $e_{1q} \otimes e_{2s}$, for $q \ne s$, $q, s = 1, 2, \ldots$. It follows then that Z is a Banach space, and since the restriction Λ of S to Z is one-to-one, continuous, and onto we have that Z isomorphic to l_r.

Clearly, Z coincides with Y. On the other hand, Y has a topological complement in X formed by all the vectors w in X whose series expansion with respect to the basis (8) has zero coefficients for the terms $e_{1n} \otimes e_{2n}$, $n = 1, 2, \ldots$. Finally, $(e_{1n} \otimes e_{2n})$ is a basis for Y equivalent to the unit basis of l_r.

We show next that the theorem holds for $m > 2$. Let us assume it is true for $m - 1$. We write

$$V := l_{p_1} \tilde{\otimes}_\pi l_{p_2} \tilde{\otimes}_\pi \cdots \tilde{\otimes}_\pi l_{p_{m-1}}.$$

We find a number $p > 1$ such that

$$\frac{1}{p_1} + \frac{1}{p_2} + \cdots + \frac{1}{p_{m-1}} = \frac{1}{p}.$$

We know then that the closed linear span W of

$$\left(e_{1n} \otimes e_{2n} \otimes \cdots \otimes e_{(m-1)n}\right) \tag{9}$$

in V is isomorphic to l_p and (9) is equivalent to the unit basis of l_p. Also, W has a topological complement D in V. Since $1/p + 1/p_m = 1/r$, we apply what we mentioned before and have that the closed linear span Y of

$$\left(e_{1n} \otimes e_{2n} \otimes \cdots \otimes e_{(m-1)n} \otimes e_{mn}\right) \tag{10}$$

in $W \tilde{\otimes}_\pi l_{p_m}$ is isomorphic to l_r and (10) is equivalent to the unit basis of l_r. Besides, Y has a topological complement in $W \tilde{\otimes}_\pi l_{p_m}$. On the other hand, $W \tilde{\otimes}_\pi l_{p_m}$ has a topological complement in X isomorphic to $D \tilde{\otimes}_\pi l_{p_m}$. The result is now immediate. Q.E.D.

COROLLARY 1. *Let* (e_{jn}) *be the unit vector basis of* l_{p_j}, $j = 1, 2, \ldots, m$, $m > 1$. *If*

$$\frac{1}{p_1} + \frac{1}{p_2} + \cdots + \frac{1}{p_m} = \frac{1}{r},$$

$r > 1$, *and* s *is the conjugate number of* r, *the following properties hold*:

(1) *For each* g *in* $\mathscr{L}(l_{p_1}, l_{p_2}, \ldots, l_{p_m})$, $Tg := (g(e_{1n}, e_{2n}, \ldots, e_{mn}))$ *belongs to* l_s *and*

$$T \colon \mathscr{L}\left(l_{p_1}, l_{p_2}, \ldots, l_{p_m}\right) \to l_s$$

is a continuous onto linear map.

(2) *For each g in $\mathscr{L}(l_{p_1}, l_{p_2}, \ldots, l_{p_m})$ and $x_j \in X_j$, $j = 1, 2, \ldots, m$, the series*

$$(Sg)(x_1, x_2, \ldots, x_m) := \sum_{n=1}^{\infty} g(e_{1n}, e_{2n}, \ldots, e_{mn})\langle x_1, e_{1n}^*\rangle\langle x_2, e_{2n}^*\rangle \cdots \langle x_m, e_{mn}^*\rangle$$

converges, Sg is in $\mathscr{L}(l_{p_1}, l_{p_2}, \ldots, l_{p_m})$, and

$$S\colon \mathscr{L}(l_{p_1}, l_{p_2}, \ldots, l_{p_m}) \to \mathscr{L}(l_{p_1}, l_{p_2}, \ldots, l_{p_m})$$

is a continuous projection whose range is isomorphic to l_s.

Proof. Let $X := l_{p_1} \tilde{\otimes}_\pi l_{p_2} \tilde{\otimes}_\pi \cdots \tilde{\otimes}_\pi l_{p_m}$, and let Y denote the closed linear subspace of X spanned by the sequence

$$(e_{1n} \otimes e_{2n} \otimes \cdots \otimes e_{mn}).$$

Let x^* be the element of X^* associated to g. Then,

$$\langle e_{1n} \otimes e_{2n} \otimes \cdots \otimes e_{mn}, x^*\rangle = g(e_{1n}, e_{2n}, \ldots, e_{mn}), \qquad n = 1, 2, \ldots,$$

and by our former theorem, we have that

$$Tg := (g(e_{1n}, e_{2n}, \ldots, e_{mn}))$$

belongs to l_s and

$$T\colon \mathscr{L}(l_{p_1}, l_{p_2}, \ldots, l_{p_m}) \to l_s$$

is linear and continuous. On the other hand, if $(a_n) \in l_s$, there is an element g^* of Y^* such that

$$\langle e_{1n} \otimes e_{2n} \otimes \cdots \otimes e_{mn}, y^*\rangle = a_n, \qquad n = 1, 2, \ldots .$$

We extend y^* to an element u of X^*. If f is the element of $\mathscr{L}(l_{p_1}, l_{p_2}, \ldots, l_{p_m})$ associated to u, it follows that $Tf = (a_n)$, which concludes the proof of (1).

As for part (2), notice that $(\langle x_j, e_{jn}^*\rangle)$ belongs to l_{p_j}, $j = 1, 2, \ldots, m$, and, hence, $(\langle x_1, e_{1n}^*\rangle\langle x_2, e_{2n}^*\rangle \cdots \langle x_m, e_{mn}^*\rangle) \in l_s$, then, since $(g(e_{1n}, e_{2n}, \ldots, e_{mn})) \in l_r$, the series $(Sg)(x_1, x_2, \ldots, x_m)$ is convergent and defines a continuous m-linear functional of $\mathscr{L}(l_{p_1}, l_{p_2}, \ldots, l_{p_m})$. It follows that S is a continuous projection with range isomorphic to l_s. Q.E.D.

The proof of the next corollary follows directly from Corollary 1.

COROLLARY 2. *Let* (e_n) *be the unit vector basis of* l_p. *If* m *is an integer,* $1 < m < p$, *and* s *is the conjugate number of* p/m, *then the following properties hold*:

(1) *For each* $P \in \mathscr{P}(^m l_p)$, $TP := (P(e_n))$ *belongs to* l_s *and the mapping*

$$T: \mathscr{P}\left(^m l_p\right) \to l_s$$

is linear continuous and onto.

(2) *For each* $P \in \mathscr{P}(^m l_p)$ *and* $x \in l_p$, *the series*

$$(SP)(x) := \sum_{n=1}^{\infty} P(e_n)\langle x, e_n^* \rangle^m$$

converges, SP *belongs to* $\mathscr{P}(^m l_p)$, *and*

$$S: \mathscr{P}\left(^m l_p\right) \to \mathscr{P}\left(^m l_p\right)$$

is a continuous projection whose range is isomorphic to l_s.

A sequence (x_n) in a Banach space X has a lower p-estimate $(1 \le p < \infty)$ if there is a constant $c > 0$ such that

$$\left\| \sum_{n=1}^{m} a_n x_n \right\| \ge c \left(\sum_{n=1}^{m} |a_n|^p \right)^{1/p}$$

for every finite sequence $a_1, a_2, \dots, a_m$ in K. If, besides, (x_n) is a basic sequence, it follows that if

$$x = \sum_{n=1}^{\infty} a_n x_n \in [x_n],$$

then (a_n) is in l_p and there is a one-to-one continuous linear map T from $[x_n]$ into l_p such that

$$Tx_n = e_n, \qquad n = 1, 2, \dots,$$

where (e_n) is the unit basis of l_p. If $p = 1$, we have that $[x_n]$ is isomorphic to l_1 and (x_n) is a Schauder basis for $[x_n]$ equivalent to the unit basis of l_1.

THEOREM 2. *In a Banach space X_j, let (x_{jn}) be a Schauder basis with a lower p_j-estimate, $j = 1, 2, \ldots, m$, such that*

$$\frac{1}{p_1} + \frac{1}{p_2} + \cdots + \frac{1}{p_m} \geq 1.$$

If W is the closed linear span of

$$(x_{1n} \otimes x_{2n} \otimes \cdots \otimes x_{mn}) \tag{11}$$

in $X := X_1 \tilde{\otimes}_\pi X_2 \tilde{\otimes}_\pi \cdots \tilde{\otimes}_\pi X_m$, we have that W is isomorphic to l_1, has a topological complement and (11) *is a Schauder basis of W equivalent to the unit basis of l_1.*

Proof. If $p \geq p_1$, clearly the basis (x_{1n}) has a lower p-estimate. Consequently, it means no restriction to assume that

$$\frac{1}{p_1} + \frac{1}{p_2} + \cdots + \frac{1}{p_m} = 1.$$

Let T_j be the continuous linear injection of X_j into l_{p_j} such that $T_j x_{jn} = e_{jn}$, $n = 1, 2, \ldots,$ where (e_{jn}) is the unit basis of l_{p_j}, $j = 1, 2, \ldots, m$. If

$$Z := l_{p_1} \tilde{\otimes}_\pi l_{p_2} \tilde{\otimes}_\pi \cdots \tilde{\otimes}_\pi l_{p_m},$$

let

$$S: X \to Z$$

be the continuous linear map such that

$$S(x_1 \otimes x_2 \otimes \cdots \otimes x_m) = T_1 x_1 \otimes T_2 x_2 \otimes \cdots \otimes T_m x_m,$$
$$x_j \in X_j,\ j = 1, 2, \ldots, m,$$

that is, $S := T_1 \otimes T_2 \otimes \cdots \otimes T_m$. Let Y denote the closed linear hull of the sequence

$$(e_{1n} \otimes e_{2n} \otimes \cdots \otimes e_{mn}) \tag{12}$$

in Z. By Theorem 1, Y is isomorphic to l_1, has a topological complement, and (12) is a Schauder basis for Y equivalent to the unit basis of l_1. By [9], (11) is a Schauder basis of W. Again by [9], the set

$$\{x_{1n_1} \otimes x_{2n_2} \otimes \cdots \otimes x_{mn_m} : n_1, n_2, \ldots, n_m = 1, 2, \ldots\} \tag{13}$$

may be ordered into a sequence (u_n) so that it is a Schauder basis for X and with (11) as a subsequence. Similarly, we provide the set

$$\{e_{1n_1} \otimes e_{2n_2} \otimes \cdots \otimes e_{mn_m} \colon n_1, n_2, \ldots, n_m = 1, 2, \ldots\}$$

with the same ordering as that of (13) and thus obtain a Schauder basis (v_n) of Z such that (12) is a subsequence of (v_n) and $Su_n = v_n$, $n = 1, 2 \ldots$. Take now an arbitrary element

$$x := \sum_{n=1}^{\infty} a_n u_n \tag{14}$$

in X. Then

$$Sx = \sum_{n=1}^{\infty} a_n v_n \tag{15}$$

is in Z. If b_n is the coefficient of $x_{1n} \otimes x_{2n} \otimes \cdots \otimes x_{mn}$ in (14), it follows that b_n is also the coefficient of $e_{1n} \otimes c_{2n} \otimes \cdots \otimes e_{mn}$ in (15), $n = 1, 2, \ldots,$ and having in mind the proof of Theorem 1, we have that $(b_n) \in l_1$. Therefore, W is formed by all the vectors (14) such that the coefficient of $x_{1n_1} \otimes x_{2n_2} \otimes \cdots \otimes x_{mn_m}$ is zero when the number $n_1, n_2, \ldots, n_m$ are not all equal. Then, W has a topological complement in X made of all those vectors of the form (14) whose coefficients for $x_{1n} \otimes x_{2n} \otimes \cdots \otimes x_{mn}$, $n = 1, 2, \ldots,$ are all zero. Q.E.D.

COROLLARY 3. *In a Banach space X_j, let (x_{jn}) be a Schauder basis with a lower p_j-estimate, $j = 1, 2, \ldots, m$, with*

$$\frac{1}{p_1} + \frac{1}{p_2} + \cdots + \frac{1}{p_m} \geq 1.$$

The following properties hold:

(1) *If, for each g in $\mathscr{L}(X_1, X_2, \ldots, X_m)$, we put $Tg := (g(x_{1n}, x_{2n}, \ldots, x_{mn}))$, then*

$$T \colon \mathscr{L}(X_1, X_2, \ldots, X_m) \to l_\infty$$

is an onto continuous linear map.

(2) *For each g in $\mathscr{L}(X_1, X_2, \ldots, X_m)$ and $x_j \in X_j$, $j = 1, 2, \ldots, m$, the series*

$$(Sg)(x_1, x_2, \ldots, x_m) := \sum_{n=1}^{\infty} g(x_{1n}, x_{2n}, \ldots, x_{mn}) \langle x_1, x_{1n}^* \rangle \langle x_2, x_{2n}^* \rangle \ldots \langle x_m, x_{mn}^* \rangle$$

converges, Sg belongs to $\mathscr{L}(X_1, X_2, \ldots, X_m)$*, and*

$$S: \mathscr{L}(X_1, X_2, \ldots, X_m) \mapsto \mathscr{L}(X_1, X_2, \ldots, X_m)$$

is a continuous projection of range isomorphic to l_∞.

Proof. This is totally analogous to that of Corollary 2, just making use of Theorem 2 instead of Theorem 1. Q.E.D.

The proof of the following corollary follows directly after Corollary 3.

COROLLARY 4. *In a Banach space* X*, let* (x_n) *be a Schauder basis with a lower* p*-estimate. If* m *is an integer with* $m \geq p$*, the following properties hold*:

(1) *If, for each* P *in* $\mathscr{P}(^m X)$*, we define* $TP := (Px_n)$*, then*

$$T: \mathscr{P}(^m X) \to l_\infty$$

is an onto continuous linear map.

(2) *For each* P *in* $\mathscr{P}(^m X)$ *and* x *in* X*, the series*

$$(SP)(x) := \sum_{n=1}^{\infty} P(x_n)\langle x, x_n^* \rangle^m$$

converges, SP belongs to $\mathscr{P}(^m X)$*, and*

$$S: \mathscr{P}(^m X) \to \mathscr{P}(^m X)$$

is a continuous projection whose range is isomorphic to l_∞.

PROPOSITION 1. *Let* (x_{jn}) *be a seminormalized sequence in* l_{p_j} *weakly convergent to zero,* $j = 1, 2, \ldots, m$*. If*

$$\frac{1}{p_1} + \frac{1}{p_2} + \cdots + \frac{1}{p_m} = \frac{1}{r}$$

with $r \geq 1$*, there is a sequence of positive integers,* $n_1 < n_2 < \cdots < n_s < \cdots$*, such that the closed linear subspace* Z *spanned by*

$$\left(x_{1n_s} \otimes x_{2n_s} \otimes \cdots \otimes x_{mn_s}\right) \tag{16}$$

in $X := l_{p_1} \tilde{\otimes}_\pi l_{p_2} \tilde{\otimes}_\pi \cdots \tilde{\otimes}_\pi l_{p_m}$ *is isomorphic to* l_r*, has a topological complement, and* (16) *is a Schauder basis for* Z *equivalent to the unit basis of* l_r.

Proof. From a result of Pelczynski [12], there is a sequence of integers $n_1 < n_2 < \cdots < n_s < \cdots$ such that the closed linear hull Y_j of (x_{jn_s}) in l_{p_j} is complemented, isomorphic to l_{p_j}, and (x_{jn_s}) is a basis for Y_j equivalent to the unit vector basis of l_{p_j}, $j = 1, 2, \ldots, m$. If Z denotes the closed linear hull of

$$\left(x_{1n_s} \otimes x_{2n_s} \otimes \cdots \otimes x_{mn_s}\right)$$

in

$$Y := Y_1 \tilde{\otimes}_\pi Y_2 \tilde{\otimes}_\pi \cdots \tilde{\otimes}_\pi Y_m,$$

then the result follows after Theorem 1, having in mind that Y has a topological complement in X. Q.E.D.

Let f be a continuous m-linear functional defined on $l_{p_1} \times l_{p_2} \times \cdots \times l_{p_m}$ with

$$\frac{1}{p_1} + \frac{1}{p_2} + \cdots + \frac{1}{p_m} = \frac{1}{r}, \qquad r > 1.$$

Making use of Proposition 1, we see next that f is weakly sequentially continuous. Taking a seminormalized sequence (x_{jn}) in l_{p_j}, weakly convergent to zero, we show that $f(x_{1n}, x_{2n}, \ldots, x_{mn})$ converges to zero. If not, we apply the Bessaga−Pelczynski selection theorem [5] and, after Proposition 1, we obtain a sequence of positive integers such that

$$\left(f(x_{1n_s}, x_{2n_s}, \ldots, x_{mn_s})\right)$$

does not converge to zero, the closed linear span Z of

$$\left(x_{1n_s} \otimes x_{2n_s} \otimes \cdots \otimes x_{mn_s}\right) \tag{17}$$

in $X := l_{p_1} \tilde{\otimes}_\pi l_{p_2} \tilde{\otimes}_\pi \cdots \tilde{\otimes}_\pi l_{p_m}$ is isomorphic to l_r, and (17) is a Schauder basis for Z equivalent to the unit basis of l_r. Hence, (17) converges weakly to the origin in X. If g is the element of X^* associated to f, we have that

$$\lim_s f(x_{1n_s}, x_{2n_s}, \ldots, x_{mn_s}) = \lim_s g\left(x_{1n_s} \otimes x_{2n_s} \otimes \cdots \otimes x_{mn_s}\right) = 0,$$

which is a contradiction.

The previous result can be found in [14] for $m = 2$. Now, if T is a continuous m-linear map from X into l_q, where

$$\frac{1}{p_1} + \frac{1}{p_2} + \cdots + \frac{1}{p_m} < \frac{1}{q},$$

let p_{m+1} be the conjugate number of q. Then

$$\frac{1}{p_1} + \frac{1}{p_2} + \cdots + \frac{1}{p_m} + \frac{1}{p_{m+1}} < 1$$

and, therefore, if h denotes the continuous $(m+1)$-linear functional in $l_{p_1} \times l_{p_2} \times \cdots \times l_{p_m} \times l_{p_{m+1}}$ such that

$$h(x_1, x_2, \dots, x_{m+1}) = \langle x_{m+1}, T(x_1, x_2, \dots, x_m) \rangle,$$
$$x_j \in l_{p_j},\ j = 1, 2, \dots, m+1,$$

it follows, after what we did before, that h is weakly sequentially continuous, thus we easily obtain that T is compact [1, 13].

A given sequence (x_n) in a Banach space X is said to have an upper p-estimate $(1 \leq p < \infty)$ if there is a constant $c > 0$ such that

$$\left\| \sum_{n=1}^{m} a_n x_n \right\| \leq c \left(\sum_{n=1}^{m} |a_n|^p \right)^{1/p},$$

for every finite sequence $a_1, a_2, \dots, a_m$ in K. This is equivalent to saying that if a_n is in l_p, then $\sum_{n=1}^{\infty} a_n x_n$ converges in X and hence there is a continuous linear mapping T from l_p into X such that

$$Te_n = x_n, \qquad n = 1, 2, \dots,$$

where (e_n) is the unit vector basis of l_p. When the sequence (x_n) is basic the mapping T is then one-to-one.

PROPOSITION 2. *Let (x_{jn}) be a sequence in the Banach space X_j with an upper p_j-estimate, $j = 1, 2, \dots, m$, such that*

$$\frac{1}{p_1} + \frac{1}{p_2} + \cdots + \frac{1}{p_m} = \frac{1}{r}, \qquad r > 1.$$

Then

$$(x_{1n} \otimes x_{2n} \otimes \cdots \otimes x_{mn})$$

has an upper r-estimate in $X := X_1 \tilde{\otimes}_\pi X_2 \tilde{\otimes}_\pi \cdots \tilde{\otimes}_\pi X_m$.

Proof. Let (e_{jn}) be the unit vector basis of l_{p_j}. Let T_j be the continuous linear mapping from l_{p_j} into X_j such that

$$T_j e_{jn} = x_{jn}, \qquad j = 1, 2, \dots, m,\ n = 1, 2, \dots .$$

Let $T := T_1 \otimes T_2 \otimes \cdots \otimes T_m$. Let Z be the closed linear hull of

$$(e_{1n} \otimes e_{2n} \otimes \cdots \otimes e_{mn}) \tag{18}$$

in $l_{p_1} \tilde{\otimes}_\pi l_{p_2} \tilde{\otimes}_\pi \cdots \tilde{\otimes}_\pi l_{p_m}$. It follows that Z is isomorphic to l_r and (18) is a Schauder basis of Z equivalent to the unit basis of l_r. Consequently, if S denotes the restriction of T to Z, then

$$S: Z \to X_1 \tilde{\otimes}_\pi X_2 \tilde{\otimes}_\pi \cdots \tilde{\otimes}_\pi X_m$$

is continuous linear and

$$S(e_{1n} \otimes e_{2n} \otimes \cdots \otimes e_{mn}) = x_{1n} \otimes x_{2n} \otimes \cdots \otimes x_{mn}, \qquad n = 1, 2, \ldots;$$

the conclusion now follows. Q.E.D.

Let (e_n) be the unit vector basis of l_p $(1 < p < \infty)$ and P an m-homogeneous continuous polynomial defined in l_p such that $1 < m < p$. If s is the conjugate number of p/m, then $(P(e_n)) \in l_s$, [15]. In [10], this result is extended in the following way: (a) *Let P be an m-homogeneous continuous polynomial from a Banach space X into a Banach space Y. If $m < p$, then P turns sequences with an upper p-estimate in X into sequences with an upper (p/m) estimate in Y*. Result (a) may be obtained after our Proposition 2.

Take a sequence (x_n) in X with an upper p-estimate. We apply Proposition 2 and thus obtain a continuous linear map T from $l_{(p/m)}$ into v$L := X \tilde{\otimes}_\pi X \tilde{\otimes}_\pi \cdots \tilde{\otimes}_\pi X$ such that

$$T(f_n) = x_n \otimes x_n \otimes \overset{(m)}{\cdots} \otimes x_n, \qquad n = 1, 2, \ldots,$$

where (f_n) is the unit basis of $l_{(p/m)}$. Let f be the symmetric m-linear map from ${}^m X$ into Y associated to P. Let $\aleph$ be the canonical mapping of ${}^m X$ into L and let g be the continuous linear map from L into Y such that $g \circ \aleph = f$. Then $S := g \circ T$ is a continuous linear map from $l_{(p/m)}$ into Y and

$$S(f_n) = g\left(x_n \otimes x_n \otimes \overset{(m)}{\cdots} \otimes x_n\right) = P(x_n), \qquad n = 1, 2, \ldots;$$

the result now follows.

Let $0 \leq \alpha < 1$. We say, after Pelczynski [13], that a sequence (x_n) in the Banach space X τ_α-converges to the origin if there is a constant $c > 0$ such that

$$\Big\| \sum_{n \in B} x_n \Big\| \le c|B|^{\alpha}$$

for every non-empty finite subset B of $\mathbb{N}$. As it is noticed in [1], this is equivalent to having a constant $b > 0$ such that

$$\Big\| \sum_{n \in B} \lambda_n x_n \Big\| \le b|B|^{\alpha}$$

for every non-empty finite subset B of $\mathbb{N}$ and every set $\{\lambda_n\colon n \in B\}$ of scalars with $|\lambda_n| \le 1$, $n \in B$. Notice also that τ_α-convergence is stronger than weak convergence.

PROPOSITION 3. *Let $\alpha_j \ge 0$, $j = 1, 2, \ldots, m$, with $\alpha := \sum_{j=1}^m \alpha_j < 1$. Let (x_{jn}) be a sequence in the Banach space X_j that τ_{α_j}-converges to the origin, $j = 1, 2, \ldots, m$. Then, in $X := X_1 \tilde{\otimes}_\pi X_2 \tilde{\otimes}_\pi \cdots \tilde{\otimes}_\pi X_m$, the sequence $(x_{1n} \otimes x_{2n} \otimes \cdots \otimes x_{mn})$ τ_α-converges to the origin.*

Proof. Let $c_j > 0$ be such that

$$\Big\| \sum_{n \in B} \lambda_n x_{jn} \Big\| \le c_j |B|^{\alpha_j},$$

where B is an arbitrary non-empty finite subset of $\mathbb{N}$ and $\lambda_n \in K$, $|\lambda_n| \le 1$, $n \in B$. Put $c := c_1 c_2 \ldots c_m$. We find an element f in X^*, $\|f\| = 1$, such that, for fixed B,

$$\begin{aligned}
&\Big\| \sum_{n \in B} x_{1n} \otimes x_{2n} \otimes \cdots \otimes x_{mn} \Big\| \\
&\qquad = \Big\langle \sum_{n \in B} x_{1n} \otimes x_{2n} \otimes \cdots \otimes x_{mn}, f \Big\rangle \\
&\qquad = \sum_{n \in B} g(x_{1n}, x_{2n}, \ldots, x_{mn}) \\
&\qquad \le \Big| g\Big(\sum_{n \in B} \gamma_{1n} x_{1n}, \sum_{n \in B} \gamma_{2n} x_{2n}, \ldots, \sum_{n \in B} \gamma_{mn} x_{mn} \Big) \Big| \\
&\qquad \le \Big\| \sum_{n \in B} \gamma_{1n} x_{1n} \Big\| \Big\| \sum_{n \in B} \gamma_{2n} x_{2n} \Big\| \ldots \Big\| \sum_{n \in B} \gamma_{mn} x_{mn} \Big\| \le c|B|^{\alpha}. \quad \text{Q.E.D.}
\end{aligned}$$

COROLLARY 5. *In a Banach space X_j, let (x_{jn}) be a basic sequence equivalent to the unit vector basis of c_0. Let (u_{jn}) be a bounded sequence in*

X_j^* *such that* $(x_{jn}, u_{jn})_{n\in\mathbb{N}}$ *be a biorthogonal system,* $j = 1, 2, \ldots, m$. *The following properties hold*:

(1) *In* $X := X_1 \tilde{\otimes}_\pi X_2 \tilde{\otimes}_\pi \ldots \tilde{\otimes}_\pi X_m$, *the closed linear span* Z *of*

$$(x_{1n} \otimes x_{2n} \otimes \cdots \otimes x_{mn}) \tag{19}$$

is isomorphic to c_0 *and* (19) *is a Schauder basis for* Z *equivalent to the unit vector basis of* c_0.

(2) *In* $\mathscr{L}(X_1, X_2, \ldots, X_m)$, *the closed linear span* U *of*

$$(u_{1n} \otimes u_{2n} \otimes \cdots \otimes u_{mn}) \tag{20}$$

is isomorphic to l_1, (20) *is a Schauder basis for* U *equivalent to the unit vector basis of* l_1, *and*

$$V := \{f \in \mathscr{L}(X_1, X_2, \ldots, X_m) : f(x_{1n}, x_{2n}, \ldots, x_{mn}) = 0,\ n = 1, 2, \ldots\}$$

is a topological complement of U *in* $\mathscr{L}(X_1, X_2, \ldots, X_m)$.

Proof. (1) Clearly, there is a constant $c > 0$ such that

$$\left\| \sum_{n\in B} x_{jn} \right\| \le c, \qquad j = 1, 2, \ldots, m,$$

for every non-empty finite subset B of $\mathbb{N}$. Consequently, (x_{jn}) τ_0-converges to the origin in X_j, $j = 1, 2, \ldots, m$. We apply Proposition 3 and have that (19) τ_0-converges to the origin in X. Hence, there is a constant $b > 0$ such that

$$\left\| \sum_{n\in B} \lambda_n x_{1n} \otimes x_{2n} \otimes \cdots \otimes x_{mn} \right\| \le b,$$

for every non-empty finite subset B of $\mathbb{N}$ and $\lambda_n \in K$, $|\lambda_n| \le 1$, $n \in B$. Let (a_n) be an element of c_0. For a given $\epsilon > 0$, we find a positive integer p such that $|a_n| \le \epsilon/b$, $n \ge p$. Then, for $p \le r \le q$,

$$\left\| \sum_{n=r}^{q} a_n x_{1n} \otimes x_{2n} \otimes \cdots \otimes x_{mn} \right\| \le \frac{\epsilon}{b} \left\| \sum_{n=r}^{q} \frac{b a_n}{\epsilon} x_{1n} \otimes x_{2n} \otimes \cdots \otimes x_{mn} \right\| \le \epsilon,$$

whence

$$\sum_{n=1}^{\infty} a_n x_{1n} \otimes x_{2n} \otimes \cdots \otimes x_{mn}$$

converges in X and thus Z is isomorphic to c_0 with (19) being a Schauder basis for Z equivalent to the unit basis of c_0.

(2) We identify, in the usual fashion, X^* with $\mathscr{L}(X_1, X_2, \ldots, X_m)$. Then V coincides with the subspace of X^* orthogonal to Z, and hence X^*/Z is isomorphic to l_1. If ψ denotes the canonical mapping of X^* onto X^*/Z,

$$\big(\psi(u_{1n} \otimes u_{2n} \otimes \cdots \otimes u_{mn})\big)$$

is a Schauder basis of X^*/Z which is equivalent to the unit basis of l_1. The result is now immediate. Q.E.D.

COROLLARY 6. *Let (e_n) be the unit vector basis of c_0. Given an integer $m > 1$, we have that in $\mathscr{P}({}^m c_0)$ the closed linear subspace U spanned by the sequence*

$$(e_n^* \otimes e_n^* \otimes \cdots \otimes e_n^*) \tag{21}$$

is isomorphic to l_1, (21) *is a Schauder basis for U equivalent to the unit basis of l_1, and*

$$V := \left\{P \in \mathscr{P}\left({}^m c_0\right) : P(e_n) = 0, n = 1, 2, \ldots\right\}$$

is a topological complement of U in $\mathscr{P}({}^m c_0)$.

Proof. This follows immediately after our previous corollary by identifying, in the usual way, $\mathscr{P}({}^m c_0)$ with $\mathscr{L}_s({}^m c_0)$. Q.E.D.

Let (e_n) be the unit vector basis of c_0 and m an integer greater than one. It is shown in [2] that if $P \in \mathscr{P}({}^m c_0)$ and $TP := (P(e_n))$, then $(P(e_n)) \in l_1$ and

$$T : \mathscr{P}\left({}^m c_0\right) \to l_1$$

is an onto continuous linear map.

This result can be obtained after Corollary 6 in the following manner: Take $P \in \mathscr{P}({}^m c_0)$. Then $P = P_1 + P_2$, $P_1 \in U$, $P_2 \in V$,

$$P_1 = \sum_{n=1}^{\infty} a_n e_n^* \otimes e_n^* \otimes \overset{(m)}{\cdots} \otimes e_n^*, \qquad (a_n) \in l_1,$$

and thus

$$\big(P(e_n)\big) = \big(P_1(e_n)\big) = (a_n) \in l_1.$$

On the other hand, if $(b_n) \in l_1$, the polynomial

$$Q := \sum_{n=1}^{\infty} b_n e_n^* \otimes e_n^* \otimes \cdots \otimes e_n^*$$

belongs to U and it satisfies $(Q(e_n)) = (b_n)$.

The following result can be found in [1] and, particularly for polynomials in [13, 3]: (b) *Let* $\alpha_j \geq 0$, $j = 1, 2, \ldots, m$, *with* $\alpha := \sum_{j=1}^{m} \alpha_j < 1$. *Let* (x_{jn}) *be a sequence in the Banach space* X_j *which* τ_{α_j}*-converges to the origin,* $j = 1, 2, \ldots, m$. *If* f *is a continuous* m*-linear mapping from* $X_1 \times X_2 \times \cdots \times X_m$ *into the Banach space* Y, *then* $(f(x_{1n}, x_{2n}, \ldots, x_{mn}))$ τ_α*-converges to the origin in* Y.

Result (b) may be obtained after Proposition 3: In $X := X_1 \tilde{\otimes}_\pi X_2 \tilde{\otimes}_\pi \cdots \tilde{\otimes}_\pi X_m$, $(x_{1n} \otimes x_{2n} \otimes \cdots \otimes x_{mn})$ τ_α-converges to the origin. Let g be the linear map from X to Y associated to f. Then, for a given non-empty finite subset B of $\mathbb{N}$, we have

$$\begin{aligned}
\left\| \sum_{n \in B} f(x_{1n}, x_{2n}, \ldots, x_{mn}) \right\| &= \left\| \sum_{n \in B} g(x_{1n} \otimes x_{2n} \otimes \cdots \otimes x_{mn}) \right\| \\
&= \left\| g\left(\sum_{n \in B} x_{1n} \otimes x_{2n} \otimes \cdots \otimes x_{mn} \right) \right\| \\
&\leq \left\| \sum_{n \in B} x_{1n} \otimes x_{2n} \otimes \cdots \otimes x_{mn} \right\|
\end{aligned}$$

and the result follows.

LEMMA 2. *In a Banach space* X, *let* (x_n) *be a seminormalized unconditional basic sequence weakly convergent to zero. If there is a continuous* m-homogeneous *polynomial* P *defined in* X *such that* $(P(x_n))$ *does not converge to zero, then there is a subsequence* (y_n) *of* (x_n) *which has a lower* m-estimate.

Proof. We may assume that $\|P\| = 1$ and, by conveniently multiplying x_n by a certain modulus-one scalar, that $P(x_n)$ is a real number greater than a positive number a^m, $n = 1, 2, \ldots$. Suppose first that every continuous homogeneous polynomial Q on X, of degree less than m, satisfies

$$\lim_n Q(x_n) = 0.$$

Let f be the symmetric m-linear functional associated to P. Then, if $m_1, m_2, \ldots, m_r$, $1 \leq r < m$, are positive integers, we have that

$$\lim_n f(x_{m_1}, x_{m_2}, \ldots, x_{m_r}, x_n, x_n, \ldots, x_n) = 0.$$

We set $n_1 := 1$, $y_1 := x_{n_1}$. Proceeding by recurrence, we assume to have already have found a positive integer n_r. Let $y_r := x_{n_r}$. We take $n_{r+1} > N - r$ such that if $p_1, p_2, \ldots, p_m$ are arbitrary positive integers not all equal being n_{r+1} the greatest of all of them, then

$$\left| f(x_{p_1}, x_{p_2}, \ldots, x_{p_m}) \right| < \frac{1}{2^{r+1}(r+1)^m}.$$

We put $y_{r+1} := x_{n_{r+1}}$. Now, take

$$y := \sum_{n=1}^{\infty} b_n y_n \in [y_n].$$

Let k be the unconditional constant of the sequence (y_n). Given $\epsilon > 0$, we find an integer n_0 such that if $n_0 \le p \le q$,

$$\left\| \sum_{n=p}^{q} b_n y_n \right\| < \frac{a\epsilon}{4k}, \qquad \frac{1}{2^{n_0}} < \frac{a^m \epsilon^m}{2}, \qquad |b_n| \le 1, n \ge n_0,$$

then,

$$\begin{aligned}
\left(\frac{a\epsilon}{4k}\right)^m &\ge \left\| \sum_{n=p}^{q} b_n y_n \right\|^m \ge (2k)^{-m} \left\| \sum_{n=p}^{q} |b_n| y_n \right\|^m \\
&\ge (2k)^{-m} \left| P\left(\sum_{n=p}^{q} |b_n| y_n \right) \right| \\
&= (2k)^{-m} \left| f\left(\sum_{n=p}^{q} |b_n| y_n, \ldots, \sum_{n=p}^{q} |b_n| y_n \right) \right| \\
&\ge (2k)^{-m} \left(\sum_{n=p}^{q} |b_n|^m P(y_n) - \left(\frac{1}{2^{p+1}} + \frac{1}{2^{p+2}} + \cdots + \frac{1}{2^q} \right) \right) \\
&\ge (2k)^{-m} \left(\sum_{n=p}^{q} |b_b|^m a^m - \frac{a^m \epsilon^m}{2} \right)
\end{aligned}$$

and

$$\left(\sum_{n=p}^{q} |b_n| \right)^{1/m} < \epsilon.$$

Hence, $(b_n) \in l_m$. We now consider the linear map T from $[y_n]$ into l_m such that if $y := \sum_{n=1}^{\infty}$, $Ty = (b_n)$. It is easy to see that T has closed graph and, consequently, T is continuous. Therefore, (y_n) has a lower m-estimate.

Suppose now that there is a positive integer $r < m$ and a continuous r-homogeneous polynomial Q on X such that $(Q(x_n))$ does not converge to zero and each continuous homogeneous polynomial R on X of degree less than r satisfies

$$\lim_n R(x_n) = 0.$$

Then, using a similar argument to the one used before, replacing P by Q, there is a subsequence (y_n) of (x_n) and a continuous linear mapping S from $[y_n]$ into l_r such that $Ty_n = e_{rn}$, $n = 1, 2 \ldots$, where (e_{rn}) is the unit basis of l_r. If (e_n) is the unit basis of l_m, there is a continuous linear map $\varphi \colon l_r \to l_m$ such that $\varphi(e_{rn}) = e_n$, $n = 1, 2, \ldots$. If we put $T := \varphi \circ S$, then $T \colon [y_n] \to l_m$ is continuous, linear, and $Ty_n = e_n$, $n = 1, 2, \ldots$. Q.E.D.

THEOREM 3. *In a Banach space X, let (x_n) be a seminormalized sequence weakly convergent to the origin. Let P be a continuous m-homogeneous polynomial defined in X such that $(P(x_n))$ does not converge to zero. Then there is a subsequence (z_n) of (x_n) and a constant $c > 0$ such that*

$$\left\| \sum_{n \in B} z_n^* \right\| \leq a |B|^{(m-1)/m}$$

for every non-empty finite subset B of $\mathbb{N}$.

Proof. After [4, Chap. III], one can find a subsequence (u_n) of (x_n) such that, for any given subsequence (w_n) of (u_n), (w_n) has a spreading model $(G, (f_n)), (f_n) = (w_n)$, such that (w_n^*) has in $[w_n]^*$ a spreading model which can be identified with $(H, (f_n^*))$, where H denotes the closed linear hull of (f_n^*) in G^*.

Let us assume first that every continuous homogeneous polynomial Q on X of degree less than m satisfies $\lim_n Q(x_n) = 0$. We denote by f the symmetric m-linear functional associated to P. We may assume, by conveniently multiplying x_n by modulus-one scalars, $n = 1, 2, \ldots$, that $P(x_n)$ is real, $\lim_n P(x_n) = c > 0$, and

$$c - \frac{1}{2^{n+1}} < P(u_n) < c + \frac{1}{2^{n+1}}, \qquad n = 1, 2, \ldots.$$

We may also assume, by selecting the elements u_n step by step, $n = 1, 2, \ldots$, as we did before in the proof of Lemma 2 in order to choose (y_n), that if $n_1, n_2, \ldots, n_m$ are positive integers not all equal with r being the greatest of them, then

$$\left| f(u_{n_1}, u_{n_2}, \ldots, u_{n_m}) \right| < \frac{1}{2^r r^m}.$$

Let $(F,(e_n))$ be the spreading model of $(u_n),(u_n)=(e_n)$. We take an element of F

$$a_1 e_{r_1} + a_2 e_{r_2} + \cdots + a_s e_{r_s}, \qquad r_1 < r_2 < \cdots < r_s,$$

of norm one. If δ is the supremum of the norm of e_n^* in F^* for all n, it follows that $|a_j| \le \delta$, $j = 1, 2, \ldots, s$. We may find, after [6], a positive integer $k > r_s$ such that

$$\left| \left\| \sum_{j=1}^{s} a_j e_{k+j} \right\|_F - \left\| \sum_{j=1}^{s} a_j u_{k+j} \right\| \right| < 1.$$

Then

$$\begin{aligned}
&\left| P\left(\sum_{j=1}^{s} a_j e_{r_j} \right) \right| \\
&\qquad = \left| \sum_{1 \le n_1, n_2, \ldots, n_m \le s} a_{n_1} a_{n_2} \ldots a_{n_m} f(u_{r_{n_1}}, u_{r_{n_2}}, \ldots, u_{r_{n_m}}) \right| \\
&\qquad \le \left| \sum_{j=1}^{s} a_j^m P(u_{r_j}) \right| + \delta^m \left(r_2^m \frac{1}{2^{r_2} r_2^m} + r_3^m \frac{1}{2^{r_3} r_3^m} + \cdots + r_s^m \frac{1}{2^{r_s} r_s^m} \right) \\
&\qquad \le \left| \sum_{j=1}^{s} a_j^m P(u_{k+j}) \right| + \left| \sum_{j=1}^{s} a_j^m \left(P(u_{r_j}) - P(u_{k+j}) \right) \right| + \delta^m \\
&\qquad \le \left| \sum_{j=1}^{s} a_j^m P(u_{k+j}) \right| + \delta^m \sum_{j=1}^{s} \frac{1}{2^{r_j}} + \delta^m \\
&\qquad \le \left| P\left(\sum_{j=1}^{s} a_j u_{k+j} \right) \right| + \delta^m \left((k+2)^m \frac{1}{2^{k+2}(k+2)^m} \right. \\
&\qquad\quad \left. + (k+3)^m \frac{1}{2^{k+3}(k+3)^m} + \cdots + (k+s)^m \frac{1}{2^{k+s}(k+s)^m} \right) + 2\delta^m \\
&\qquad \le \left| P\left(\sum_{j=1}^{s} a_j u_{k+j} \right) \right| + 3\delta^m \le \|P\| \cdot \left| \sum_{j=1}^{s} a_j u_{k+j} \right| + 3\delta^m \\
&\qquad \le \|P\| \cdot \left| \left\| \sum_{j=1}^{s} a_j u_{k+j} \right\| - \left\| \sum_{j=1}^{s} a_j e_{k+j} \right\|_F \right| \\
&\qquad\quad + \|P\| \cdot \left\| \sum_{j=1}^{s} a_j u_{k+j} \right\|_F + 3\delta^m \\
&\qquad \le 2\|P\| + 3\delta^m,
\end{aligned}$$

and thus P is a continuous m-homogeneous polynomial in the linear subspace of F generated by (e_n). We extend P to a continuous m-homogeneous polynomial S on F. If (e_n) does not converge weakly to the origin in F, then F is isomorphic to l_1 and (e_n) is a Schauder basis of F equivalent to the unit basis of l_1 [4, Chap. I]. Hence, (e_n) has a lower m-estimate.

If, on the contrary, (e_n) converges weakly to zero in F, then (e_n) is an unconditional basis of this space [7; 4, Chap. I]. We now apply Lemma 2 and obtain a subsequence (e_{n_j}) of (e_n) with a lower m-estimate. Let $v_j := u_{n_j}$, $j = 1, 2, \dots$. Then, (v_j) has a spreading model $(L, (g_j))$, with $g_j = e_{n_j}$, $j = 1, 2$, and L being the closed linear span of (e_{n_j}) in F. If we denote by M the closed linear hull of (g_j^*) in L^*, it follows that $(M, (g_j^*))$ is a spreading model of (v_n^*) in $[v_n]^*$. There is a constant $b > 0$ such that if B is any non-empty finite subset of $\mathbb{N}$ and $b_n \in K$, $n \in B$, then

$$\left\| \sum_{n \in B} b_n g_n \right\| \geq b \left(\sum_{n \in B} |b_n|^m \right)^{1/m}.$$

For fixed B, we find $\sum_{n \in B} c_n g_n$ of norm one in L such that

$$\left\| \sum_{n \in B} g_n^* \right\| < 2 \left| \left\langle \sum_{n=1}^{\infty} c_n g_n, \sum_{n \in B} g_n^* \right\rangle \right|.$$

If k is the unconditional constant of (g_n), we have that

$$\left\| \sum_{n=1}^{\infty} |c_n| g_n \right\| \leq 2k \left\| \sum_{n=1}^{\infty} c_n x_n \right\|$$

and therefore

$$\begin{aligned} \left\| \sum_{n \in B} g_n^* \right\| &< 2 \left| \left\langle \sum_{n=1}^{\infty} c_n g_n, \sum_{n \in B} g_n^* \right\rangle \right| = 2 \left| \sum_{n \in B} c_n \right| \\ &\leq 2 \sum_{n \in B} |c_n| \leq 2 \left(\sum_{n=1}^{\infty} |c_n|^m \right)^{1/m} |B|^{(m-1)/m} \\ &\leq 4kb^{-1} \left\| \sum_{n=1}^{\infty} c_n x_n \right\| \cdot |B|^{(m-1)/m} = 4kb^{-1} |B|^{(m-1)/m}. \end{aligned}$$

Now we apply [1] and obtain a subsequence $(v_{m_n}^*)$ of (v_n^*) in $[v_n]^*$ and a constant $a > 0$ such that

$$\left\| \sum_{n \in B} v_{m_n}^* \right\| \leq a|B|^{(m-1)/m}$$

for every non-empty fintie subset B of $\mathbb{N}$. If we now set $z_n := v_{m_n}$, $n = 1, 2, \ldots,$ it follows immediately that in $[Z_n]^*$ we have

$$\left\| \sum_{n \in B} z_n^* \right\| \leq a|B|^{(m-1)/m}. \tag{22}$$

Suppose now that there is a positive integer $r < m$ and a continuous r-homogeneous polynomial Q defined in X such that $(Q(x_n))$ does not converge to zero and, nevertheless, each continuous s-homogeneous polynomial T in X, $s < r$, satisfies that $\lim_n T(x_n) = 0$. Applying to Q what we did before, we obtain a subsequence (z_n) of (x_n) and $a > 0$ such that

$$\left\| \sum_{n \in B} z_n^* \right\| \leq a|B|^{(r-1)/r} \leq a|B|^{(m-1)/m}$$

for every non-empty finite subset B of $\mathbb{N}$. Q.E.D.

Note. In a Banach space X, let (x_n) be a basic sequence weakly convergent to zero. Let P be a continuous m-homogeneous polynomial defined in X such that $(P(x_n))$ does not converge to zero. We have seen in our previous theorem that there is a subsequence (z_n) of (x_n) and there is a positive number a for which (22) is satisfied. For $p > m$, we apply [8] and obtain that (z_n^*) has an upper $(p/(p-1))$-estimate, thus, by duality, (z_n) has a lower p-estimate. However, when m is an even integer this result can be improved as we show in the following.

THEOREM 4. *In a Banach space X, let (x_n) be a seminormalized sequence weakly convergent to the origin. Let P be a continuous m-homogeneous polynomial defined in X such that $(P(x_n))$ does not converge to zero. Then, if m is even, there is a subsequence (y_n) of (x_n) with a lower m-estimate.*

Proof. Without loss of generality, we may assume $\|P\| = 1$, $\|x_n\| = 1$, $P(x_n) > 0$, $\lim_n P(x_n) = c > 0$,

$$c - \frac{1}{2^{n+1}} < P(x_n) < c + \frac{1}{2^{n+1}}, \qquad n = 1, 2, \ldots .$$

Let g be the symmetric m-linear functional defined in X^m associated to P. Let us supopse first that every continuous r-homogeneous polynomial Q

defined in X, with $r < m$, satisfies that $\lim_n Q(x_n) = 0$. We now select step by step, similarly to what was done in the proof of Lemma 2, a subsequence (y_n) of (x_n) such that if $n_1, n_2, \ldots, n_m$ are positive integers not all equal and r is the greatest of them

$$\left|g(y_{n_1}, y_{n_2}, \ldots, y_{n_m})\right| \leq \frac{1}{2^r r^m}.$$

Let (a_n) be a sequence of real numbers such that the series $\sum_{n=1}^{\infty} a_n y_n$ converges in X. For a given $\epsilon > 0$, we find a positive integer n_0 such that

$$\left\|\sum_{n=p}^{q} a_n y_n\right\| < \epsilon, \quad |a_n| < \epsilon, \quad n_0 \leq p \leq n \leq q, \quad c - \frac{1}{2^{n_0+1}} > \frac{1}{2}c.$$

Then, for such integers p and q,

$$\begin{aligned}
\epsilon^m &\geq \left\|\sum_{n=p}^{q} a_n y_n\right\|^m \geq \left|P\left(\sum_{n=p}^{q} a_n y_n\right)\right| \\
&= \left|\sum_{p \leq n_1, n_2, \ldots, n_m \leq q} a_{n_1} a_{n_2} \ldots a_{n_m} g(y_{n_1}, y_{n_2}, \ldots, y_{n_m})\right| \\
&\geq \left|\sum_{n=p}^{q} a_n^m P(y_n)\right| - \epsilon^m \left(\frac{1}{2^{p+1}} + \frac{1}{2^{p+2}} + \cdots + \frac{1}{2^q}\right) \\
&\geq \left|\sum_{n=p}^{q} a_n^m P(y_n)\right| - \frac{1}{2}\epsilon^m \\
&\geq \sum_{n=p}^{q} a_n^m \left(c - \frac{1}{2^{n+1}}\right) - \frac{1}{2}\epsilon^m \geq \frac{1}{2}c \sum_{n=p}^{q} a_n^m - \frac{1}{2}\epsilon^m,
\end{aligned}$$

and so

$$c \sum_{n=p}^{q} a_n^m \leq 3\epsilon^m.$$

Consequently, $(a_n) \in l_m$. If (a_n) is now a sequence in $\mathbb{C}$ such that $\sum_{n=1}^{\infty} a_n y_n$ converges in X, then, if $a_n = b_n + ic_n$, $b_n \in \mathbb{R}$, $c_n \in \mathbb{R}$, we have that $\sum_{n=1}^{\infty} b_n y_n$ and $\sum_{n=1}^{\infty} c_n y_n$ both converge in X, it follows that (b_n) and (c_n) are in l_m, and so $(a_n) \in l_m$.

Let T be the mapping from $[y_n]$ into l_m such that if $y := \sum_{n=1}^{\infty} a_n y_n \in [y_n]$, then $Ty := (a_n)$. After what we have just seen, T is well defined.

Clearly, since T has a closed graph, we have that it is continuous. Therefore, there is $a > 0$ such that

$$\left\| \sum_{n=1}^{s} a_n x_n \right\| \geq a \left(\sum_{n=1}^{s} |a_n|^m \right)^{1/m}, \qquad a_1, a_2, \ldots, a_s \in K. \tag{23}$$

Now, if on the other hand there is a continuous r-homogeneous polynomial Q in X, with $r < m$, such that $(Q(x_n))$ does not converge to zero, one can make use of the former Note also obtaining (23). Q.E.D.

Problem. Does Theorem 4 still hold for odd m?

THEOREM 5. *In a Banach space X_j, let (x_{jn}) be a seminormalized basic sequence weakly convergent to the origin, $j = 1, 2, \ldots, m$. Let (f_{jn}) be bounded sequence in X_j^* such that $(x_{jn}, f_{jn})_{n \in \mathbb{N}}$ is a biorthogonal system. If (f_{jn}) has an upper p_j-estimate with*

$$\frac{p_1 - 1}{p_1} + \frac{p_2 - 1}{p_2} + \cdots + \frac{p_m - 1}{p_m} \geq 1,$$

the following properties hold:

(1) *In $X := X_1 \tilde{\otimes}_\pi X_2 \tilde{\otimes}_\pi \cdots \tilde{\otimes}_\pi X_m$, the closed linear span Z of*

$$(x_{1n} \otimes x_{2n} \otimes \cdots \otimes x_{mn}) \tag{24}$$

is complemented, it is isomorphic to l_1, and (24) *is a Schauder basis for Z equivalent to the unit basis of l_1.*

(2) *If, for each continuous m-linear functional g defined in $X_1 \times X_2 \times \cdots \times X_m$, we put $Tg := (g(x_{1n}, x_{2n}, \ldots, x_{mn}))$, then*

$$T \colon \mathscr{L}(X_1, X_2, \ldots, X_m) \to l_\infty$$

is an onto continuous linear map.

(3) *For each g in $\mathscr{L}(x_1, x_2, \ldots, X_m)$ and $x_j \in X_j$, $j = 1, 2, \ldots, m$, the series*

$$(Sg)(x_1, x_2, \ldots, x_m)$$
$$:= \sum_{n=1}^{\infty} g(x_{1n}, x_{2n}, \ldots, x_{mn}) f_{1n}(x_1) f_{2n}(x_2) \ldots f_{mn}(x_m)$$

converges, Sg belongs to $\mathscr{L}(X_1, X_2, \ldots, X_m)$, and

$$S \colon \mathscr{L}(X_1, X_2, \ldots, X_m) \to \mathscr{L}(X_1, X_2, \ldots, X_m)$$

is a continuous projection of range isomorphic to l_∞.

Proof. For $j = 1, 2, \ldots, m$, we write $q_j = p_j/(p_j - 1)$, (e_{jn}) is the unit vector basis of l_{q_j}, T_j is the mapping from l_{p_j} into X_j^* such that

$$T_j((a_n)) = \sum_{n=1}^{\infty} a_n f_{jn}, \qquad (a_n) \in l_{p_j},$$

and let

$$S_j \colon X_j \to l_{q_j}$$

be the conjugate map of T_j. Clearly, $S_j x_{jn} = e_{jn}$, $n = 1, 2, \ldots$. Let ϕ be the mapping from X into $L := l_{q_1} \tilde{\otimes}_\pi l_{q_2} \tilde{\otimes}_\pi \ldots \tilde{\otimes}_\pi l_{q_m}$ given by the tensor product of the maps $S_1, S_2, \ldots, S_m$. Let Y be the closed linear subspace of L generated by

$$(e_{1n} \otimes e_{2n} \otimes \cdots \otimes e_{mn}). \tag{25}$$

By Theorem 2, Y is complemented, isomorphic to l_1, and (25) is a Schauder basis for Y equivalent to the unit basis of l_1. Let φ be a continuous projection of L onto Y. Then $\psi := \varphi \circ \phi$ is a continuous linear map from X onto Y such that

$$\psi(x_{1n} \otimes x_{2n} \otimes \cdots \otimes x_{mn}) = e_{1n} \otimes e_{2n} \otimes \cdots \otimes e_{mn}, \qquad n = 1, 2, \ldots .$$

Consequently, Z is isomprhic to l_1, it is complemented in X, and (25) is a Schauder basis for Z equivalent to the unit basis of l_1. This proves part (1).

Obviously, if g belongs to $\mathscr{L}(X_1, X_2, \ldots, X_m)$, $(g(x_{1n}, x_{2n}, \ldots, x_{mn}))$ is in l_∞ and

$$T \colon \mathscr{L}(X_1, X_2, \ldots, X_m) \to l_\infty$$

is linear and continuous. On the other hand, given (a_n) in l_∞, there is a continuous linear functional u on Z such that

$$u(x_{1n} \otimes x_{2n} \otimes \cdots \otimes x_{mn}) = a_n, \qquad n = 1, 2, \ldots .$$

The Hahn–Banach theorem allows us to extend u to an element v of X^*. If f denotes the element of $\mathscr{L}(X_1, X_2, \ldots, X_m)$ associated to v, then

$$f(x_{1n}, x_{2n}, \ldots, x_{mn}) = v(x_{1n} \otimes x_{2n} \otimes \cdots \otimes x_{mn}) = a_n, \qquad n = 1, 2, \ldots,$$

so T is onto, and part (2) is thus proved.

If $x_j \in X_j$, then $(f_{jn}(x_j))$ belongs to l_{q_j} and hence, using Hölder's inequality, $(f_{1n}(x_1), f_{2n}(x_2), \ldots, f_{mn}(x_m))$ is in l_1. Thus, if $g \in \mathscr{L}(X_1, X_2, \ldots, X_m)$,

$$\begin{aligned}(Sg)&(x_1, x_2, \ldots, x_m) \\ &:= \sum_{n=1}^{\infty} g(x_{1n}, x_{2n}, \ldots, x_{mn}) f_{1n}(x_1) f_{2n}(x_2) \ldots f_n(x_m)\end{aligned}$$

defines an element Sg of $\mathscr{L}(X_1, X_2, \ldots, X_m)$ such that $S^2 g = Sg$. We then have that

$$S : \mathscr{L}(X_1, X_2, \ldots, X_m) \to \mathscr{L}(X_1, X_2, \ldots, X_m)$$

is a continuous projection. On the other hand, the mapping Λ form the range of S into l_∞ such that

$$\Lambda(Sg) = (g(x_{1n}, x_{2n}, \ldots, x_{mn})), \qquad g \in \mathscr{L}(X_1, X_2, \ldots, X_m),$$

is an isomorphism. Hence, the range of S is isomorphic to l_∞. Q.E.D.

COROLLARY 7. *Let X be a Banach space such that X^* has type $p \in (1, 2]$. Let (x_n) be a seminormalized basic sequence in X weakly convergent to the origin. If m is na integer greater than $p/(p - 1)$ there exist a subsequence (y_n) of (x_n) and a basic sequence (f_n) in X^*, with $(x_n, f_n)_{n \in \mathbb{N}}$ beig a biorthogonal system, such that the following properties hold*:

(1) *If, for each $P \in \mathscr{P}(^m X)$, we define $T(P) := (P(y_n))$, then*

$$T : \mathscr{P}(^m X) \to l_\infty$$

is an onto continuous linear map.

(2) *For each P in $\mathscr{P}(^m X)$ and each x in X, the series*

$$S(P)(x) := \sum_{n=1}^{\infty} P(y_n) f_n(x)^m$$

converges, $S(P)$ belongs to $\mathscr{P}(^m X)$, and

$$S : \mathscr{P}(^m X) \to \mathscr{P}(^m X)$$

is a continuous projection of range isomorphic to l_∞.

Proof. Since $p > m/(m - 1)$, we proceed as in [8, Theorem 3.5] and find a subsequence (y_n) of (x_n) and a sequence (f_n) in X^* such that $(y_n, f_n)_{n \in \mathbb{N}}$ is a biorthogonal system and (f_n) has an upper $(m/(m - 1))$-estimate. The result follows now directly after the previous theorem. Q.E.D.

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